International Journal of Theoretical Physics, Vol. 42, No. 6, June 2003 (© 2003)

Activation-Like Processes at Zero Temperature

Daniel Arteaga,[1,4] Esteban Calzetta,[2] Albert Roura,[3] and Enric Verdaguer[1]

Received April 14, 2003

We examine the possibility that a metastable quantum state could experiment a phenomenon similar to thermal activation but at zero temperature. To do that we study the real-time dynamics of the reduced Wigner function in a simple open quantum system: an anharmonic oscillator with a cubic potential linearly interacting with an environment of harmonic oscillators. Our results suggest that this activation-like phenomenon exists indeed as a consequence of the fluctuations induced by the environment and that its associated decay rate is comparable to the tunneling rate as computed by the instanton method, at least for the particular potential of the system and the distribution of frequencies for the environment considered in this paper. However, we are not able to properly deal with the term which leads to tunneling in closed quantum systems, and a definite conclusion cannot be reached until tunneling and activation-like effects are considered simultaneously.

KEY WORDS: thermal activation; quantum tunneling; open quantum systems.

1. INTRODUCTION

The study of the decay rate of a state trapped in a metastable state by a potential barrier has a long and distinguished history in both statistical physics and quantum mechanics. In statistical mechanics one is usually worried about the thermal activation effect, by which a particle escapes over a potential barrier due to the fluctuations induced by a thermal bath. As a paradigm we have the classical work by Kramers (1940), who considered a classical Brownian particle trapped in a metastable minimum and computed the escape probability by analyzing the dynamics of its probability distribution function, in both the underdamped and

[1] Departament de Física Fonamental and CER en Astrofísica, Física de Partícules i Cosmologia, Universitat de Barcelona, Barcelona, Spain.

[2] Departamento de Física, Universidad de Buenos Aires, Ciudad Universitaria, Buenos Aires, Argentina.

[3] Department of Physics, University of Maryland, College Park, Maryland.

[4] To whom correspondence should be addressed at Departament de Física Fonamental and CER en Astrofísica, Física de Partícules i Cosmologia, Universitat de Barcelona, Avenue Diagonal 647, 08028 Barcelona, Spain; e-mail: darteaga@ffn.ub.es.

0020-7748/03/0600-1257/0 © 2003 Plenum Publishing Corporation

overdamped cases. In quantum mechanics one is interested in the tunneling effect, by which a particle escapes the local minimum by traversing the potential barrier through a classically forbidden region. A technique which has been of great success to compute the tunneling rate in quantum mechanics and quantum field theory is the instanton method (Callan and Coleman, 1977; Coleman, 1977, 1985), where the decay probability can be computed in terms of the classical trajectories in imaginary time. In both thermal activation and tunneling the decay rate r, meaning the decay probability per unit time, follows an approximate exponential law, $r \approx A \exp(-B)$. In thermal activation, $B = V_s/(kT)$, where k is Boltzmann's constant, T is the absolute temperature, and V_s is the height of the free energy measured from the metastable minimum; in tunneling $B = S_E/\hbar$ where S_E is the action for a suitable trajectory which goes under the barrier in imaginary time.

In recent years mesoscopic physics has become a center of attention. Experimental advances are pushing the boundaries between classical and quantum systems and lead, in particular, to the possibility of observing quantum tunneling for systems that can be described by macroscopic variables. These are essentially open quantum systems, which are characterized by a distinguished subsystem within a larger closed quantum system, described by some degrees of freedom which are subject to physical experimentation, and the rest of the system, described by generally unobservable degrees of freedom which act as an environment or a bath for the distinguished subsystem. The environment induces both dissipation and noise to the distinguished system, which is usually referred to as the "system" for short. Many of these open quantum systems can be equivalently represented by a particle subject to an arbitrary external potential and coupled to an environment consisting of an infinite set of independent harmonic oscillators. A number of known physical systems can be modeled by adjusting the coupling of the system and environment variables and choosing appropriate potentials.

Caldeira and Leggett (1981, 1983b), in two influential papers, considered the effect of an environment on quantum tunneling. They were able to generalize the instanton method to a simple open quantum system. In particular, they considered an anharmonic quadratic plus cubic potential bilinearly coupled an environment at zero temperature consisting of an infinite set of harmonic oscillators, with frequencies distributed according to the so-called ohmic distribution. They argued that this system is a very good model for the flux trapped in a superconducting quantum interference device (SQUID), a single Josephson junction biased by a fixed external current, and others. Assuming that the environment degrees of freedom are only weakly perturbed by the interaction with the system, they concluded that dissipation always tends to suppress tunneling.

Fujikawa *et al.* (1992a,b) reanalyzed the same problem as Caldeira and Leggett employing canonical perturbation theory and the second quantization formalism for the system instead of the instanton method, with a quadratic plus quartic potential to have a well-defined ground state. By considering the two lowest energy

eigenstates of the system, they were able to reproduce Caldeira and Leggett's results. They studied the effect of the next two excited states and concluded that, once these states are taken into account, dissipation can enhance tunneling for some distributions of frequencies of the environment, but not for the ohmic distribution considered by Caldeira and Leggett.

In some recent papers a different real-time approach to compute the vacuum decay rate in quantum field theory was introduced (Calzetta *et al.*, 2001, 2002) as a first step to consider situations far from the equilibrium. The analysis is based on the time evolution equation, the so-called master equation, for the reduced Wigner function describing the open quantum system. The quantum field theory problem was reduced to an open quantum system described by a single degree of freedom associated to the modes of the field which are nearly homogeneous within a region whose size corresponds to a nucleating bubble of true vacuum, and coupled to an infinite set of harmonic oscillators corresponding to the inhomogeneous modes of the field. The coupling was linear in the system variables but quadratic in the environment ones. The master equation contained dissipation and noise terms, which describe the influence of the environment on the system, as well as derivative terms in the momentum coordinates (third order for a cubic potential), which are responsible for quantum tunneling in closed quantum systems.

Unfortunately it is not possible to compute the total decay rate in a closed analytic or semianalytic form because it is very difficult to deal with the third-order derivative terms. One should resort to methods such as those based on matrix continued fractions to compute the decay rate master equations with third-order derivative terms (Risken and Vogel, 1988). However in the work by Calzetta *et al.* (2001, 2002), as well as in the present paper, one was primarily interested in the contribution from the environment backreaction on the vacuum decay rate and, hence, the rather drastic approximation of neglecting the third derivative term responsible for tunneling was made. The decay rate obtained was entirely due to the terms which at high temperature are responsible for thermal activation. In principle this approximation should be correct provided that the timescales associated to the activation-like effect and the tunneling effect are very different. The remarkable result was that the activation-like effect produced by the backreaction of the inhomogeneous modes was, in fact, larger than the tunneling effect obtained with the instanton method.

The fact that activation may be important as a backreaction effect can be understood by noting that the characteristic timescale for the decay process is much larger than the dynamical and relaxation timescales (see Calzetta and Verdaguer, 1999, for a detailed analysis). In fact, although the effects of dissipation and noise are very small on the characteristic dynamical timescale, they can have a cumulative effect which becomes important in the long run. The real-time approach based on the master equation seems a suitable technique to deal with those backreaction effects, but is difficult to implement when addressing the tunneling effect. On the

other hand, the instanton method is very well suited for studying the tunneling effect, but, at least in some cases, seems to underplay the backreaction of the environment on the system.

The aim of this paper is to study whether the results obtained by Calzetta *et al.* (2001, 2002) are a particular feature of the particular system that they considered or, on the contrary, a general feature which can be extended to a wider class of systems. Therefore, using the real-time techniques mentioned earlier, we reanalyze the simpler model studied by Caldeira and Leggett (1981, 1983b) to check whether a similar activation-like effect is also found in that case.

There are several differences between the model considered in this paper and the field theory case analyzed by Calzetta *et al.* (2001, 2002). First, we consider here a bilinear coupling between the system and environment, whereas the coupling considered in the field theory case was quadratic in the environment degrees of freedom. Second, in the field theory case both the spectral distribution of the environment frequencies and the value of the coupling constant were a priori determined by the particular system–environment separation considered there, whereas we shall freely choose the spectral distribution and the coupling parameter. Third, we will employ several techniques (harmonic approximation, Kramers method, and lowest eigenvalue expansion) which could not be applied to the field theory problem because of the particular features of that model.

The plan of the paper is the following. In Section 2 we set up the model. In Section 3, we present the master equation for the reduced Wigner function. Then, following closely Calzetta *et al.* (2002), we neglect the third-order derivative term and concentrate on the weak dissipation case by studying the averaged dynamics over an oscillation period. In Section 4 we obtain an analytical expression for the decay rate by assuming a harmonic approximation for the classical trajectories. Two alternative approaches are employed. The first one is based on a perturbative expansion for the lowest eigenvalue, whereas the second one is based on Kramers's classical work. Finally, in Section 5 we discuss the implications of our result.

2. THE OPEN QUANTUM SYSTEM MODEL

Let us consider a particle of mass M, the "system," subject to an arbitrary potential $V(x)$ and coupled to a bath of independent harmonic oscillators of mass m, the "environment." Let us assume that the system and the environment are linearly coupled. The action for the whole set of degrees of freedom is given by

$$S[x, \{q_j\}] = S_{\rm s}[x] + S_{\rm e}[\{q_j\}] + S_{\rm int}[x, \{q_j\}], \tag{1a}$$

where the terms on the right-hand side, which correspond to the action of the system, the environment, and the interaction term respectively, are given by

$$S_{\rm s}[x] = \int dt\, \left(\frac{1}{2}M\dot{x}^2 - V(x)\right), \tag{1b}$$

$$S_{\rm e}[\{q_j\}] = \sum_j \int dt\, \left(\frac{1}{2}m\dot{q}_j^2 - \frac{1}{2}m\omega_j^2 q_j^2\right), \tag{1c}$$

$$S_{\rm int}[x, \{q_j\}] = \sum_j c_j \int dt\, x(t) q_j(t), \tag{1d}$$

with c_j being the system–environment coupling parameters and ω_j the environment oscillator frequencies. The system potential $V(x)$ includes a quadratic part, corresponding to an oscillator of frequency Ω_0, and an anharmonic part $V^{(\rm nl)}$,

$$V(x) = \frac{1}{2}\Omega_0^2 x^2 + V^{(\rm nl)}(x). \tag{2}$$

At this point, the potential $V^{(\rm nl)}(x)$ is arbitrary, but later on we will take a cubic potential $V^{(\rm nl)}(x) = -(\lambda/6)x^3$. It will be convenient for us to rewrite the interaction term as

$$S_{\rm int}[x, \{q_j\}] = \int_0^\infty \frac{2m\omega}{\pi c(\omega)} I(\omega) \int dt\, x(t) q(t;\omega), \tag{3}$$

where $c(\omega)$ and $q(t;\omega)$ are functions such that $c(\omega_j) = c_j$ and $q(t;\omega_j) = q_j(t)$ and

$$I(\omega) = \sum_j \frac{\pi c_j^2}{2m\omega_j}\delta(\omega - \omega_j) \tag{4}$$

is the spectral density of the environment.

When the system and the environment are initially uncorrelated, i.e., when the initial density matrix factorizes, the evolution for the reduced density matrix can be written as

$$\rho_{\rm r}(x, x', t) = \int dx_{\rm i}\, dx_{\rm i}'\, J(x, x', t; x_{\rm i}, x_{\rm i}', t_{\rm i})\rho_{\rm r}(x_{\rm i}, x_{\rm i}', t_{\rm i}), \tag{5}$$

where the propagator J is found to be, in a path integral representation,

$$J(x_{\rm f}, x_{\rm f}', t; x_{\rm i}, x_{\rm i}', t_{\rm i}) = \int_{x(t_{\rm i})=x_{\rm i}}^{x(t)=x_{\rm f}} \mathcal{D}x \int_{x'(t_{\rm i})=x_{\rm i}'}^{x'(t)=x_{\rm f}'} \mathcal{D}x'\, e^{i(S[x]-S[x']+S_{\rm IF}[x,x'])/\hbar}, \tag{6}$$

where $S_{\rm IF}[x, x']$ is the influence action, related to the influence functional $F_{\rm IF}$ introduced by Feynman and Vernon (1963) through $F_{\rm IF}[x, x'] = \exp(i S_{\rm IF}[x, x']/\hbar)$. For a Gaussian initial density matrix for the environment, the influence action can be expressed as (Caldeira and Leggett, 1983a; Feynman and Hibbs, 1965; Feynman

and Vernon, 1963):

$$S_{\rm IF}[x,x'] = -2\int_{t_{\rm i}}^{t} ds \int_{t_{\rm i}}^{s} ds'\,\Delta(s)D(s,s')\Sigma(s') + \frac{i}{2}\int_{t_{\rm i}}^{t} ds \int_{t_{\rm i}}^{t} ds'\,\Delta(s)N(s,s')\Delta(s'), \tag{7}$$

where $\Sigma \equiv (x+x')/2$ and $\Delta \equiv x'-x$. The kernels $D(t,t')$ and $N(t,t')$ are called the dissipation and noise kernels, respectively. If initially we are at thermal equilibrium at a temperature T, these kernels are given by

$$D(t,t') = \int_0^\infty \frac{d\omega}{\pi} I(\omega)\,\sin\omega(t-t'), \tag{8a}$$

$$N(t,t') = \int_0^\infty \frac{d\omega}{\pi} I(\omega)\,\coth\left(\frac{\hbar\omega}{2kT}\right)\cos\omega(t-t'). \tag{8b}$$

The influence action can be divergent and a renormalization procedure may be required, as can be seen by reexpressing the influence action as

$$S_{\rm IF}[x,x'] = \int_{t_{\rm i}}^{t} ds \int_{t_{\rm i}}^{s} ds'\,\Delta(s)H(s,s')\Sigma(s') + \frac{i}{2}\int_{t_{\rm i}}^{t} ds \int_{t_{\rm i}}^{t} ds'\,\Delta(s)N(s,s')\Delta(s'), \tag{9}$$

where, at least formally, $H(t,t') \equiv -2\theta(t-t')D(t,t')$, being $\theta(t-t')$ the step function. The kernel $H(t,t')$ is a product of two distributions, which in general is not well defined and may contain divergences. Nevertheless, it is always possible to introduce suitable counterterms in the bare frequency of the system Ω_0 in order to compensate the divergent terms coming from $H(t,t')$ (see Caldeira and Leggett, 1983a; Calzetta *et al.*, 2003; Roura and Verdaguer, 1999, for more details). However in the particular problem in which we are interested, this issue will turn to be unimportant since the divergent parts of the kernel $H(t,t')$ will cancel in the final results. Thus, we can use the bare kernel $H(t,t')$ (with some implicit regularization) instead of its renormalized expression.

Following Caldeira and Leggett (1981, 1983b), we shall consider the case of zero temperature and ohmic environment, in which we have a continuum of harmonic oscillators in the environment distributed according to

$$I(\omega) = \eta\omega. \tag{10}$$

With this spectral density the expectation value of $\hat{x}(t)$ obeys the equation of motion of a classical damped oscillator with a friction coefficient given by the proportionality constant η. In this case dissipation and noise kernels are

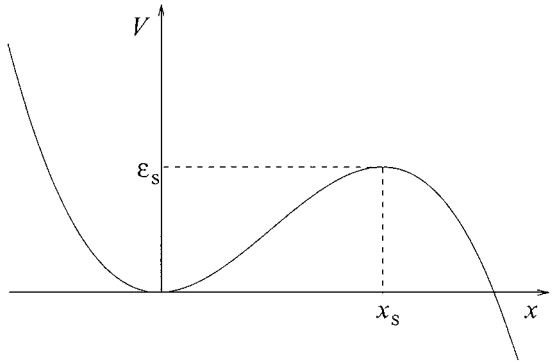

Fig. 1. Plot of the potential $V(x)$ under which the particle is confined. The maximum of the potential barrier is at x_s and corresponds to an escape energy ε_s, which we consider much larger than the zero-point energy of the harmonic oscillator, $\hbar\Omega_0/2$.

found to be

$$D(t, t') = \eta\delta'(t - t'), \tag{11a}$$

$$N(t, t) = \frac{\eta}{\pi}\mathrm{Pf}\frac{-1}{(t - t')^2}, \tag{11b}$$

where Pf indicates the Hadamard finite part prescription (Schwartz, 1957). Later on we will need the expressions of the Fourier transforms of the noise and dissipation kernels,

$$D(t, t') = \int \frac{d\omega}{2\pi}\, e^{-i\omega(t,t')}\tilde{D}(\omega), \tag{12a}$$

$$N(t, t') = \int \frac{d\omega}{2\pi}\, e^{-i\omega(t,t')}\tilde{N}(\omega), \tag{12b}$$

which in the case of zero temperature and ohmic environment are given by

$$\tilde{D}(\omega) = i\eta\omega, \qquad \tilde{N}(\omega) = \eta|\omega|. \tag{13}$$

We will concentrate on the potential $V(x) = (1/2)M\Omega_0^2x^2 - (\lambda/6)x^2$, which exhibits a metastable minimum at $x = 0$ and an unstable maximum at $x = x_s = 2M\Omega_0^2/\lambda$, which corresponds to an energy $\varepsilon_s = v(x_s) = 2M^3\Omega_0^6/(3\lambda^2)$ (see Fig. 1). The system state will be peaked located around the metastable minimum at $x = 0$, and can escape through the potential barrier. We will consider that once the particle exits the potential well region, it never reenters. Since this potential is not bounded from below, it should be understood as an approximation to a more realistic situation in which there exists an absolute minimum, located at a much

lower energy, so that the return probability is negligible. We will restrict to the situation in which the energy barrier is much larger than $\hbar\Omega_0$.

3. PHASE-SPACE DYNAMICS

3.1. Evolution of the Reduced Wigner Function

The reduced Wigner function $W_{\rm r}$ is a phase space distribution defined from the reduced density matrix $\rho_{\rm r}$ by the following integral transform:

$$W_{\rm r}(x,\,p,\,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} d\Delta\, e^{ip\Delta/\hbar}\rho_{\rm r}(x-\Delta/2,\,x+\Delta/2,\,t). \tag{14}$$

The Wigner function is a quantum mechanical analogue of a phase-space probability distribution (Hillery *et al.*, 1984; Wigner, 1932). The partial distribution $\int_{-\infty}^{\infty} dp W_{\rm r}(x,\,p,\,t)$ gives the probability density of finding the system at the position x; in the same way, $\int_{-\infty}^{\infty} dx\, W_{\rm r}(x,\,p,\,t)$ gives the probability density of finding the system with momentum p. However, the uncertainty principle prevents us from determining at the same time the position and the momentum of a particle, so that the Wigner function cannot be interpreted as a true probability density in phase space. In fact, the Wigner function is not necessarily positive-defined everywhere and in general it may acquire negative values.

Up to terms of order c_j^2, $\hbar c_j^2$, and $\hbar^2$, with c_j being the environment coupling constants, the reduced Wigner function $W_{\rm r} = W_{\rm r}(x,\,p,\,t)$ obeys the following evolution equation (Calzetta *et al.*, 2002; Roura, 2001), which we shall call master equation:

$$\frac{\partial W_{\rm r}}{\partial t} = \{H_{\rm s},\,W_{\rm r}\}_{\rm PB} + \frac{\partial}{\partial p}(\mathcal{D}W_{\rm r} + \hbar\{\mathcal{N},\,W_{\rm r}\}_{\rm PB}) - \hbar^2\frac{\lambda}{24}\frac{\partial^3 W_{\rm r}}{\partial p^3}, \tag{15}$$

where $\{\cdot,\,\cdot\}_{\rm PB}$ are the Poisson brackets,

$$\{f,\,g\}_{\rm PB} = \frac{\partial f}{dx}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial g}{\partial x},$$

H_s is the reduced system Hamiltonian,

$$H_{\rm s} = \frac{p^2}{2M} + \frac{1}{2}M\Omega_0^2 x^2 - \frac{\lambda}{6}x^3, \tag{16}$$

and $\mathcal{D}$ and $\mathcal{N}$ are given by

$$\mathcal{D}(t) = -2\int_{t_{\rm i}}^{t} dt'\, D(t,\,t')X(t';\,x,\,p), \tag{17a}$$

$$\mathcal{N}(t) = \int_{t_{\rm i}}^{t} dt'\, N(t,\,t')X(t';\,x,\,p), \tag{17b}$$

where $X(t'; x, p)$ is a solution of the classical equations of motion associated to the Hamiltonian H_s,

$$\frac{dX}{dt'} = \frac{P}{M}, \tag{18a}$$

$$\frac{dP}{dt'} = -M\Omega_0^2 X + \lambda \frac{X^2}{2}, \tag{18b}$$

with final conditions $X(t) = x$ and $P(t) = p$.

For the case of ohmic environment the friction coefficient η is proportional to c_j^2, and the above approximation for the evolution equation will be valid and consistent when the condition $S \gg \hbar \gtrsim S_\gamma / \Omega_0$ is fulfilled, where $\gamma = \eta/(2M)$ is the characteristic dissipation frequency and S is the typical action for the process considered. If we are interested in studying the jump over a potential barrier of height ε_s, this condition can be restated as $\varepsilon_s \gg \hbar\Omega_0 \gtrsim \varepsilon_s \gamma / \Omega_0$, which shows that the master equation will be valid for small dissipation and large energy barriers.

At this point it is worth making a comment on the notation. Throughout this paper we will use lowercase letters (x, p, θ, j ...) to indicate phase-space variables, which are the arguments of phase-space distributions such as the Wigner function, whereas the corresponding uppercase letters (X, P, Θ, J ...) will indicate time functions which give the phase-space position of a particle in a given time.

To introduce the Fourier transform of the coefficients $\mathcal{D}$ and $\mathcal{N}$ later on, it will be convenient for us to replace the integration limit in Eqs. (17a) and (17b) by $-\infty$, although the initial conditions are set up at $t_i = 0$. For the coefficient $\mathcal{D}$ this does not introduce any error, since the dissipation kernel only has support in $t = t'$. For the coefficient $\mathcal{N}$, this introduces a small error, which can be estimated by performing the integral in Eq. (17a), choosing a periodic function for $X(t')$ of frequency Ω_0; this will be enough in the context of the adiabatic approximation, which we will introduce later on. The result of the calculation shows that for times t, which verify $t \gg \Omega_0^{-1}$, the contribution of the integral from $-\infty$ to 0 is comparatively small. Since at the end we shall be interested in studying timescales of the order of the decay time, which are much larger than the characteristic dissipation time γ^{-1}, which will in turn be much larger than the time of oscillation Ω_0^{-1}, the approximation can be considered safe.

If the system were isolated, Eq. (15) would reduce to

$$\frac{\partial W}{\partial t} = \{H_s, W\}_{PB} - \hbar^2 \frac{\lambda}{24} \frac{\partial^2 W}{\partial p^3}, \tag{19}$$

where W is the Wigner function of the closed system. This equation is exactly equivalent to von Neumann's equation for the density matrix of a one-dimensional quantum mechanical system with the potential $V(x)$. If the last term in this equation were not present, the evolution of the Wigner function would be entirely equivalent

to that of a classical ensemble in the phase space. Hence, the term with the third derivatives must be the responsible for tunneling. In principle, one could compute the tunneling amplitude from Eq. (19), but in practice tunneling is more easily calculated with the WKB approximation to the Schrödinger equation (see, e.g., Galindo and Pascual, 1991; Landau and Lifshitz, 1977) or the instanton method (Callan and Coleman, 1977; Coleman, 1977, 1985). As we have discussed, in this contribution we are not going to deal with tunneling, but rather to compute the effect due to activation. With this aim, we shall neglect the last term of Eq. (15), which is responsible for tunneling. Hence we will use the following equation for the distribution function W_r:

$$\frac{\partial W_{\rm r}}{\partial t} = \{H_{\rm s}, W_{\rm r}\}_{\rm PB} + \frac{\partial}{\partial p}(\mathcal{D}W_{\rm r} + \hbar\{\mathcal{N}, W_{\rm r}\}_{\rm PB}). \tag{20}$$

Formally Eq. (20) can be thought as a Fokker–Planck equation, describing the dynamics of an ensemble of points in the phase space. The dynamics of this ensemble of points can be equally characterized by means of the following Langevin equation:

$$\dot{X}_{\rm s} = \frac{P_{\rm s}}{M}, \tag{21a}$$

$$\begin{aligned}\dot{P}_{\rm s} &= -V'(X_{\rm s}) - \int_{-\infty}^{\infty} dt'\, H(t,t')X_{\rm s}(t') + \xi \\ &= -M\Omega_0^2 X_{\rm s} - \eta\dot{X}_{\rm s} + \lambda\frac{X_{\rm s}^2}{2} + \xi,\end{aligned} \tag{21b}$$

where ξ is a Gaussian noise of zero mean and correlation function

$$\langle\xi(t)\xi(t')\rangle_\xi = \hbar N(t,t'), \tag{21c}$$

$X_{\rm s} = X_{\rm s}(t,\xi]$ and $P_{\rm s} = P_{\rm s}(t,\xi]$ are the stochastic functions corresponding to the phase-space variables x and p, respectively, and the second equality in Eq. (21b) is just valid in the case of ohmic environment. This Langevin equation does not describe actual trajectories of the system (meaning a continuous sequence of projectors for the position and the momentum of the system at each instant of time, which would violate Heisenberg's uncertainty principle) but must be regarded as a formal computational tool, in the same way as the Wigner function does not correspond to a true probability density. In fact, Langevin-like equations appear naturally in the context of open quantum systems when trying to derive the dynamics of the reduced Wigner function from the path integrals in Eq. (5) (Calzetta *et al.*, 2003).

3.2. Action-Angle Variables

To obtain explicit expressions for the coefficients $\mathcal{D}$ and $\mathcal{N}$, we need to solve the set of Eqs. (18), which describe the motion of a classical particle in the potential $V(x)$. Since we are interested in the motion inside the potential well, which is periodic, it is possible to introduce action-angle variables θ and j (see, e.g., Goldstein, 1980). The action variable is defined by

$$j = \frac{1}{2\pi} \oint P\, dX, \tag{22}$$

whereas the angle variable θ changes from zero to 2π and is canonically conjugate to j. Recall that the reduced system Hamiltonian can be entirely written in terms of the action variable, $H_s = H_s(j)$.

We shall consider θ and j phase-space variables like x and p, and we will analyze the Fokker-Planck equation in terms of these new variables. However, the solution of Eqs. (18) can be also described by giving the trajectory of the particle in the $\theta - j$ space. In a completely analogous way to X and P, we will consider the functions $\Theta(t'; \theta, j)$ and $J(t'; \theta, j)$, which give the angular position and action of a particle satisfying the set of Eqs. (18), with final conditions $\Theta(t) = \theta$ and $J(t) = j$. The functions Θ and J satisfy the equations of motion $\dot{\Theta} = \omega(J)$ and $\dot{J} = 0$, where $\omega(j) = dH_s(j)/d_j$ is the frequency of oscillation. With the aforementioned boundary conditions, these equations of motion can be immediately solved to give $\Theta(t'; \theta, j) = \theta + \Omega(j)(t' - t)$ and $J(t'; \theta, j) = j$.

Since θ is an angle, the transformation equation $x = x(\theta, j)$ is periodic in θ, $x(\theta, j) = x(\theta + 2\pi, j)$ and thus it can be decomposed in terms of a Fourier series with respect to θ,

$$x(\theta, j) = \sum_{n=-\infty}^{\infty} e^{in\theta} x_n(j), \tag{23}$$

where $x_{-n}(j) = x_n^*(j)$ since x is real. The trajectory of the particle can be also decomposed in terms of the Fourier series associated to the angular coordinate:

$$\begin{aligned} X(t'; \theta, j) &= \sum_{n=-\infty}^{\infty} e^{in\Theta(t'; \theta, j)} x_n(J(t'; \theta, j)) \\ &= \sum_{n=-\infty}^{\infty} e^{in[\theta + \Omega_0(t' - t)]} x_n(j). \end{aligned} \tag{24}$$

Then we can write the functions $\mathcal{D}(t)$ and $\mathcal{N}(t)$ appearing in Eq. (20) as

$$\mathcal{D}(t) = -2 \int_{-\infty}^{t} dt'\, D(t, t') X(t') = \sum_{n=-\infty}^{\infty} e^{in\theta} x_n(j) D_n(j), \tag{25a}$$

$$\mathcal{N}(t) = \int_{-\infty}^{t} dt'\, N(t,t')X(t') = \sum_{n=-\infty}^{\infty} e^{in\theta} x_n(j) N_n(j), \tag{25b}$$

where $D_n(j)$ and $N_n(j)$ are given by

$$D_n(j) = \int \frac{d\omega}{2\pi} \frac{2i\tilde{D}(\omega)}{\omega + n\Omega(j) - i\epsilon}, \tag{26a}$$

$$N_n(j) = \int \frac{d\omega}{2\pi} \frac{-i\tilde{N}(\omega)}{\omega + n\Omega(j) - i\epsilon}. \tag{26b}$$

To derive these last expressions, we have made use of the following equality: $\int_0^\infty \exp(isu) = i/(s + i\epsilon)$, where ϵ is an arbitrarily small positive real number.

3.3. Weak Dissipation Limit: Averaging Over Angles

As discussed by Kramers (1940), in the case of small dissipation, i.e., $\gamma \ll \Omega_0$, the phase-space dynamics will mostly correspond to a gradual change of the distribution of the ensemble over the different energy values. The change of the Wigner function over an oscillation period will be small, so that we may suppose that the reduced Wigner function only depends on the action variable j (or the energy E), and does not depend on the angular variable θ, $W_{\mathrm{r}}(\theta, j) = F(j)$. Thus, we can obtain a simpler equation by averaging all the terms of the Fokker-Planck equation over the variable θ. Furthermore, in this case the averaged Wigner function F is a partial distribution, and hence it admits a true probabilistic interpretation, as opposed to the nonaveraged Wigner function.

Notice that in this case $\{H_{\mathrm{s}}, F\}_{\mathrm{PB}} = 0$ and that, for any phase-space function $\Psi = \Psi(\theta, j)$,

$$\oint \frac{d\theta}{2\pi} \frac{\partial \Psi}{\partial p} = \oint \frac{d\theta}{2\pi} \{x, \Psi\}_{\mathrm{PB}} = \oint \frac{d\theta}{2\pi} \left(\frac{\partial x}{\partial \theta}\frac{\partial \Psi}{\partial j} - \frac{\partial x}{\partial j}\frac{\partial \Psi}{\partial \theta} \right)$$
$$= \oint \frac{d\theta}{2\pi} \left(\frac{\partial x}{\partial \theta}\frac{\partial \Psi}{\partial j} - \frac{\partial^2 x}{\partial j \partial \theta}\Psi \right) = \frac{d}{dj} \oint \frac{d\theta}{d\pi} \left(\frac{\partial x}{\partial \theta}\Psi \right).$$

Using the last expression, we can average Eq. (20) to obtain the following averaged Fokker-Planck equation:

$$\frac{\partial F}{\partial t} = \frac{\partial}{\partial j} \left(\hbar \bar{\mathcal{N}} \frac{\partial F}{\partial j} + \bar{\mathcal{D}} F \right), \tag{27}$$

where we have introduced $\bar{\mathcal{D}}(j)$ and $\bar{\mathcal{N}}(j)$, which are defined as follows:

$$\bar{\mathcal{D}} = \oint \frac{d\theta}{2\pi} \frac{\partial x}{\partial \theta} \mathcal{D} = -i \sum_{n=-\infty}^{\infty} |x_n(j)|^2 n D_n(j), \tag{28a}$$

$$\bar{\mathcal{N}} = \oint \frac{d\theta}{2\pi} \frac{\partial x}{\partial \theta} \frac{\partial \mathcal{N}}{\partial \theta} = \sum_{n=-\infty}^{\infty} |x_n(j)|^2 n^2 N_n(j). \tag{28b}$$

Equations (28a) and (28b) can be further simplified. Taking into account Eq. (26a) and the fact that $1/(z + i\epsilon) = P(1/z) - i\pi\delta(z)$, we can write

$$D_n(j) = \mathrm{PV} \int \frac{d\omega}{2\pi} \frac{2i\tilde{D}(\omega)}{\omega + n\Omega(j)} + \tilde{D}(n\Omega(j)), \tag{29}$$

where we took into account that $\tilde{D}(\omega)$ is an odd function. When summing over n in Eq. (28a), only the last term in Eq. (29) will contribute for every D_n. Since the contributions from the first term in D_n and D_{-n} cancel for each n because

$$\frac{D(\omega)}{\omega - n\Omega(j)} - \frac{D(\omega)}{\omega + n\Omega(j)} = \frac{2nD(\omega)}{\omega^2 - n^2\Omega^2(j)}$$

is an odd function and integrates to zero. The final result for the coefficient $\bar{\bar{D}}$ is

$$\bar{\mathcal{D}}(j) = i \sum_{n=-\infty}^{\infty} |x_n(j)|^2 n \tilde{D}(n\Omega). \tag{30a}$$

Performing similar steps with N_n, we get the following expression for $\bar{\mathcal{N}}$:

$$\bar{\mathcal{N}}(j) = \frac{1}{2} \sum_{n=-\infty}^{\infty} |x_n(j)|^2 n^2 \tilde{N}(n\Omega). \tag{30b}$$

Particularizing to the case of an ohmic environment initially at zero temperature, we have $\tilde{N}(\omega) = \eta|\omega|$, $\tilde{D}(\omega) = i\eta\omega$, which lead to

$$\bar{\mathcal{D}}(j) = 2\eta\Omega(j) \sum_{n=0}^{\infty} |x_n(j)|^2 n^2, \tag{31a}$$

$$\bar{\mathcal{N}}(j) = \eta\Omega(j) \sum_{n=0}^{\infty} |x_n(j)|^2 n^3. \tag{31b}$$

4. ENVIRONMENT-INDUCED DECAY RATE

Our aim in this section is to compute the environment-induced decay rate by solving the averaged Fokker-Planck equation, Eq. (27), with the appropriate boundary conditions. We begin by introducing a simplifying hypothesis.

4.1. The Harmonic Approximation

The nondissipative dynamics described by (18) is approximately harmonic for energies much lower than the escape energy ε_s. In this case the frequency of

that motion is simply Ω_0. For intermediate energies, the motion is qualitatively similar, but with a somewhat smaller frequency. It is only for energies extremely close to the escape energy ε_s that the motion is substantially different: the particle needs a very large amount of time to complete one period and hence the frequency tends to zero.

To obtain an order of magnitude estimate of the escape rate, it is legitimate to approximate the classical motion of Eq. (18) by its harmonic approximation, since this approximation is qualitatively valid for all the values of the energy except for a very small region of energies extremely close to ε_s.[5]

In the case of vacuum decay in quantum field theory studied by Calzetta *et al.* (2001, 2002) the harmonic approximation could not be introduced, since in that case there existed a frequency threshold with a value greater than Ω_0 in the dissipation, so that only Fourier modes with a frequency higher than the threshold contributed to it. Hence it was crucial to consider the fully nonlinear dynamics of the system. On the other hand, in our case the noise and dissipation kernels do not exhibit such a threshold and it is possible to introduce the harmonic approximation, which amounts to neglect the Fourier modes at higher frequency in front of the lowest ones in the solution of Eq. (18).

Neglecting the nonlinear term of the potential in Eq. (18b), the solution of the equations of motion with final conditions $X(t) = x$ and $P(t) = p$ can be written as

$$X(t') = \frac{1}{2}\left(x - \frac{ip}{M\Omega_0}\right) e^{i\Omega_0(t'-t)} + \frac{1}{2}\left(x + \frac{ip}{M\Omega_0}\right) e^{-i\Omega_0(t'-t)}, \tag{32a}$$

$$P(t') = \frac{1}{2}(iM\Omega_0 x + p)e^{i\Omega_0(t'-t)} + \frac{1}{2}(-iM\Omega_0 x + p)e^{-i\Omega_0(t'-t)}. \tag{32b}$$

The frequency of the motion is simply given by $\Omega(j) = \Omega_0$, and thus the action variable j is simply given by $j = \varepsilon/\Omega_0$, being $\varepsilon = p^2/(2M) + \Omega_0^2 x^2/2$ the energy. To determine the action variable, we compare the solution for $X(t')$ with Eq. (24), and write

$$X(t') = \frac{1}{2}\sqrt{\frac{2j}{M\Omega_0}}\, e^{i[\theta+\Omega_0(t'-t)]} + \frac{1}{2}\sqrt{\frac{2j}{M\Omega_0}}\, e^{-i[\theta+\Omega_0(t'-t)]}, \tag{33}$$

where we have identified the angle variable θ as

$$e^{i\theta} = \frac{M\Omega_0 x - ip}{\sqrt{(M\Omega_0 x)^2 + p^2}}. \tag{34}$$

[5] In fact, an exact solution of the equations of motion, together with a numerical analysis of the Fokker-Plank equation, reveals that the result that we will obtain here is not only qualitatively but also quantitatively valid within the degree of approximation we are working.

The decomposition of the x variable in terms of θ and j is given by

$$x(\theta, j) = \frac{1}{2}\sqrt{\frac{2j}{M\Omega_0}}\, e^{i\theta} + \frac{1}{2}\sqrt{\frac{2j}{M\Omega_0}}\, e^{-i\theta}, \tag{35}$$

so that

$$x_1(j) = x_{-1}(j) = \frac{1}{2}\sqrt{\frac{2j}{M\Omega_0}}, \qquad x_n = 0, \quad n \neq -1, 1. \tag{36}$$

The coefficients $\bar{\mathcal{D}}(j)$ and $\bar{\mathcal{N}}(j)$ can be therefore expressed as

$$\bar{\mathcal{D}}(j) = 2\gamma j, \qquad \bar{\mathcal{N}}(j) = \gamma j, \tag{37}$$

(we recall that $\gamma \equiv n/(2M)$), and the averaged Fokker-Planck equation may be written as

$$\frac{\partial F}{\partial t} = 2\gamma \frac{\partial}{\partial j}\left(\frac{\hbar\Omega_0}{2} j \frac{\partial F}{\partial j} + \Omega_0 j F\right), \tag{38}$$

or, equivalently, working with energies,

$$\frac{\partial F}{\partial t} = 2\gamma \frac{2}{\partial \varepsilon}\left(\varepsilon_0 \varepsilon \frac{\partial F}{\partial \varepsilon} + \varepsilon F\right), \tag{39}$$

where $\varepsilon_0 \equiv \hbar\Omega_0/2$. It can also be rewritten as a conservation equation,

$$\frac{\partial F}{\partial t} + \frac{\partial \Phi}{\partial \varepsilon} = 0, \tag{40}$$

where

$$\Phi = -2\gamma\left(\varepsilon_0 \varepsilon \frac{\partial F}{\partial \varepsilon} + \varepsilon F\right) \tag{41}$$

is the probability flux.

4.2. Escape Rate: Normal Mode Analysis

Assuming that the Fokker-Planck equation can be decomposed into a sum of normal modes,

$$F(t, \varepsilon) = \sum_r c_r\, e^{-rt}\, f_r(\varepsilon), \tag{42}$$

we get the following time-independent equation:

$$L f_r + r f_r = 0, \qquad L = 2\gamma \frac{d}{d\varepsilon}\left(\varepsilon_0 \varepsilon \frac{d}{d\varepsilon} + \varepsilon\right). \tag{43}$$

The boundary conditions of the partial differential equation are the following. First, we have assumed that the particle is removed once it arrives at the separatrix, so

that there will be no probability to find the particle at the separatrix: $f_r(\varepsilon_s) = 0$. Second, we will also assume a vanishing flux of incoming particles at $\varepsilon = 0$, i.e., $\Phi(0) = 0$, which is equivalent to demanding the finiteness of f_r and its derivative at $\varepsilon = 0$ ($f_r(0)$, $f_r'(0) < \infty$) as can be seen from Eq. (41).

The normal-mode analysis can be formulated as a standard Sturm–Liouville problem. The operator L, which may be written as

$$L = 2\gamma\, e^{-\varepsilon/\varepsilon_0}\varepsilon_0 \frac{d}{d\varepsilon}\left(e^{\varepsilon/\varepsilon_0}\varepsilon \frac{d}{d\varepsilon}\right) + 2\gamma, \tag{44}$$

is self-adjoint with the aforementioned boundary conditions and the scalar product defined by

$$(f, g) = (2\gamma)^{-1} \int_0^{\varepsilon_s} d\varepsilon\, e^{\varepsilon/\varepsilon_0} f^*(\varepsilon) g(\varepsilon). \tag{45}$$

Thus, the theory of differential equations guarantees that the eigenfunctions $f_r(\varepsilon)$ constitute a complete orthogonal set, and that the eigenvalues r are real (Courant and Hilbert, 1953). Furthermore, the operator L is negative definite, which can be seen as follows:

$$\begin{aligned}(f, Lf) &= \varepsilon_0 \int_0^{\varepsilon_s} d\varepsilon\, e^{\varepsilon/\varepsilon_0} f^*(\varepsilon) \frac{d}{d\varepsilon}\left[\varepsilon\, e^{-\varepsilon/\varepsilon_0} \frac{d}{d\varepsilon}\left(e^{\varepsilon/\varepsilon_0} f(\varepsilon)\right)\right] \\ &= -\varepsilon_0 \int_0^{\varepsilon_s} d\varepsilon\, \varepsilon\, e^{-\varepsilon/\varepsilon_0} \left|\frac{d}{d\varepsilon}\left(e^{\varepsilon/\varepsilon_0} f(\varepsilon)\right)\right|^2 < 0, \qquad f \neq 0,\end{aligned}$$

where we have integrated by parts in the last equality. This implies that the eigenvalues r are always positive, as expected.

Performing the change of variables $f_r(\varepsilon) = 2\gamma e^{-y} \bar{f}_{\bar{r}}(y)$, where $y \equiv \varepsilon/\varepsilon_0$ and $\bar{r} = r/(2\gamma)$, the differential equation $Lf_r = r f_r$ adopts the form of the Laguerre differential equation,

$$y \bar{f}_{\bar{r}}''(y) + (1 - y) \bar{f}_{\bar{r}}'(y) + \bar{r} \bar{f}_{\bar{r}}(y) = 0, \tag{46}$$

whose unique regular solution is given by

$$\bar{f}_{\bar{r}}(y) = N L_{\bar{r}}(y), \tag{47}$$

where $L_{\bar{r}}(y)$ are Laguerre functions (which reduce to the Laguerre polynomials in the case of nonnegative integer $\bar{r}$; see Gradsteyn and Ryzhik, 1980), and N is a normalization constant.

The solution we have found verifies the first of the boundary conditions, the regularity at the origin. Now we impose the second of the boundary conditions, namely $\bar{f}_{\bar{r}}(y_s) = 0$ (with $y_s \equiv \varepsilon_s/\varepsilon_0$). This boundary condition will imply a discretization on the possible values of the escape rate r. Finally, knowing the initial state $f_i(\varepsilon) = F(0, \varepsilon)$, we will be able to reconstruct the solution by computing the

coefficients c_r:

$$c_r = (f_r,\ f_{\rm i}) = N \int_0^{\varepsilon_{\rm s}} d\varepsilon\, L_{\frac{r}{2\gamma}}\left(\frac{\varepsilon}{\varepsilon_0}\right) f_{\rm i}(\varepsilon). \tag{48}$$

Unfortunately, it is not possible to determine analytically the possible values of $\bar{r}$ from the equation $L_{\bar{r}}(y_{\rm s}) = 0$. However, if the potential barrier were infinitely high, the second boundary condition would read $\lim_{y\to\infty} e^{-y} L_{\bar{r}}(y) = 0$, implying that $L_{\bar{r}}(y)$ should have, at most, a polynomial behavior at infinity, and the eigenvalues would be $\bar{r} = 0, 1, 2\ldots$, so that eigenmodes with $\bar{r} \neq 0$ would decay in a time given by γ^{-1} or shorter. However, the potential barrier corresponds to a some large but finite energy, and hence the real eigenvalues differ from those computed in the infinite barrier case by a small quantity, at least for those eigenvalues corresponding to eigenstates with characteristic energies much lower than the potential barrier. Thus, we can compute them perturbatively.

Furthermore, the relevant contribution to the decay rate will be given by the mode with the lowest eigenvalue, which will fulfill the condition $\bar{r} \ll 1$, since the remaining modes will decay in a time of order γ^{-1} at most. Thus, we proceed to compute perturbatively the lowest-order mode by expanding the Laguerre function around $\bar{r} = 0$:

$$L_{\bar{r}}(y) = 1 + \bar{r}[\ln y + \gamma_{\rm E} - {\rm Ei}(y)] + O(\bar{r}^2), \tag{49}$$

where $\gamma_{\rm E} = 0.577216\ldots$ is Euler's constant, and ${\rm Ei}(y)$ is the exponential integral defined as ${\rm Ei}(y) \equiv {\rm PV}\int_{-\infty}^{y} (e^u/u)\, du$. The expansion in Eq. (49) can be found by solving perturbatively Eq. (46) up to order $\bar{r}$, and imposing the correct boundary condition at $y = 0$.

Therefore, imposing the boundary condition $L_{\bar{r}}(y_{\rm s}) = 0$ is equivalent to demanding

$$r = 2\gamma\bar{r} \approx \frac{2\gamma}{{\rm Ei}(\varepsilon_{\rm s}/\varepsilon_0) - \ln(\varepsilon_{\rm s}/\varepsilon_0) - \gamma_{\rm E}}. \tag{50}$$

Since $\varepsilon_{\rm s}$ is much larger than ε_0, the exponential integral can be approximated by ${\rm Ei}(\varepsilon_{\rm s}/\varepsilon_0) \approx (\varepsilon_0/\varepsilon_{\rm s}) e^{\varepsilon_{\rm s}/\varepsilon_0}$, and the other two terms in the denominator become negligible in front of this one. Hence, we may approximate the lowest-order solution by

$$r \approx \frac{2\gamma\varepsilon_{\rm s}}{\varepsilon_0} \exp\left(-\frac{\varepsilon_{\rm s}}{\varepsilon_0}\right). \tag{51}$$

Equation (51), which gives the probability per unit time for a particle to jump the barrier, is our final result for the escape rate.

In the case of vacuum decay in quantum field theory studied by Calzetta *et al.* (2001, 2002) the time-independent Fokker-Planck equation had a continuous spectrum, and therefore in that case it was not possible either to perform an

eigenvalue expansion, like the one we have performed in this subsection, or to follow Kramers's method, as will be done in the next subsection.

4.3. Escape Rate: Kramers's Method

Our Fokker-Planck equation (39) is completely analogous to that found by Kramers (1940) in the classical underdamped case, once we replace kT, the temperature times Boltzmann's constant, by the zero-point energy of the harmonic oscillator $\varepsilon_0 = \hbar\Omega_0/2$.[6] Therefore, as an alternative to the previous subsection, we can apply the same method as Kramers to compute the decay rate.

Instead of imposing the correct boundary conditions, we look for the solutions of the Fokker-Planck equation with constant flux $\Phi = \Phi_0$. Since the flux can be rewritten as

$$\Phi = -2\gamma\, e^{-\varepsilon/\varepsilon_0}\varepsilon\varepsilon_0 \frac{\partial}{\partial\varepsilon}(e^{\varepsilon/\varepsilon_0}F), \tag{52}$$

these solutions will be given by

$$\begin{aligned} F(\varepsilon) &= \frac{\Phi_0}{2\gamma\varepsilon_0}\, e^{-\varepsilon/\varepsilon_0}\int_{\varepsilon}^{\varepsilon_s} d\varepsilon'\, \frac{e^{\varepsilon'/\varepsilon_0}}{\varepsilon'} \\ &= \frac{\Phi_0}{2\gamma\varepsilon_0}\, e^{-\varepsilon/\varepsilon_0}[\mathrm{Ei}(\varepsilon_s/\varepsilon_0) - \mathrm{Ei}(\varepsilon/\varepsilon_0)]. \end{aligned} \tag{53}$$

Again, we have imposed that $F(\varepsilon_s) = 0$ because we assume that when a particle arrives at the separatrix it never reenters the potential well region. Notice that the solution we have found has a logarithmic singular behavior at small energies, which takes into account the injection of a probability flux through the point $\varepsilon = 0$ necessary for the maintenance of the constant flux. However, since we expect the flux of probability to be very small, this contribution will be not very significative and will affect only the region of energies $\varepsilon \lesssim \varepsilon_0$, which are much lower than the scape energy ε_s.

We can compute the flux Φ_0 by imposing the correct normalization of the averaged Wigner function $F(\varepsilon)$, namely $\int_0^{\varepsilon_s} d\varepsilon F(\varepsilon) = 1$:

$$\frac{\Phi_0}{2\gamma\varepsilon_0}\int_0^{\varepsilon_s} d\varepsilon\, e^{-\varepsilon/\varepsilon_0}\int_{\varepsilon}^{\varepsilon_s} d\varepsilon'\, \frac{e^{\varepsilon'/\varepsilon_0}}{\varepsilon'} = 1. \tag{54}$$

The main contribution to the integral over ε' is due to those values for ε' which differ from ε_s by a quantity of order ε_0. We may replace ε' by its value at the separatrix, ε_s. Making this approximation, we can perform analytically the integrals in Eq. (54).

[6] Notice that this is only true under the harmonic approximation. Had we not neglected the cubic term in the equations of motion, our Fokker-Planck equation would not be equivalent to the one found by Kramers in the classical underdamped case.

Retaining only those terms which have an exponential factor $e^{\varepsilon_s}/\varepsilon_0$, we obtain the final result for the flux of particles:

$$\Phi = \Phi_0 \approx \frac{2\gamma\varepsilon_s}{\varepsilon_0} \exp\left(-\frac{\varepsilon_s}{\varepsilon_0}\right). \tag{55}$$

Although we have assumed that the flux is constant to solve the differential equation, this is not actually the case, and the flux at the separatrix Φ_s is proportional to the total probability of finding the particle in the potential well region: $\Phi_s(t) = P(t)\Phi_0$. Integrating the conservation Eq. (40) with respect to the energy, we easily find that the total probability decay follows the law

$$\frac{dP(t)}{dt} + P(t)\Phi_0 = 0,$$

which may be integrated to give $P(t) = e^{-\Phi_0 t}$. Hence the probability flux Φ_0 is to be identified with the escape rate r of last subsection. We see that both methods agree.

5. DISCUSSION

The study of quantum tunneling based on a real-time formulation seems crucial to address highly nonequilibrium situations in which no adiabaticity assumptions can be made. As mentioned in the Introduction, a first step in this direction was made in Calzetta *et al.* (2001, 2002), where the effect on vacuum decay in quantum field theory due to the backreaction of the short-wavelength modes was analyzed by regarding the long-wavelength modes responsible for tunneling as an open quantum system. In that case it was found that such a backreaction seemed to yield an enhancement of the decay rate. There are, however, a couple of aspects which deserve, in our opinion, a more careful analysis. The first one is the need for a suitable identification of the tunneling degrees of freedom and the corresponding implementation of a system–environment separation which leads to the reformulation of the problem in terms of an open quantum system. Second, when solving the master equation that governs the time evolution of the reduced Wigner function for the tunneling degree of freedom to obtain the vacuum decay rate, the attention was focused on the backreaction of the environment (the short-wavelength modes) and the higher derivative terms which would be uniquely responsible for tunneling if the system were isolated were neglected.

In this paper we have considered a fairly simple quantum mechanical open system, in which the system–environment separation is given beforehand and the coupling constant governing the interaction between the system and the environment can be adjusted at will, rather than being self-consistently determined, as happened to be the case for vacuum decay in field theory. Therefore, it is interesting to check whether a similar enhancement of the tunneling rate is obtained in

that simpler model and see if such an effect is generic. Furthermore, the result can be considered more robust than that in the field theory case since the number of important assumptions made is rather small. It is, thus, worthwhile to elaborate on this point and recall the different approximations employed throughout the paper to obtain the decay rate.

First of all, the master equation that we are considering in this paper can be obtained by keeping terms of order $\hbar$ and $\hbar\gamma$ at most, and restricting to values of γ, which are small enough (in particular we should take $(\gamma/\Omega_0)S \lesssim \hbar$, where S is the characteristic action of the problem), so that the terms of order γ^2 or higher can be neglected[7] (Roura, 2001). On the other hand, the master equation obtained in this way contains a term with third-order derivatives of the reduced Wigner function with respect to the momentum (the term of order $\hbar^2$). This sort of terms, which are absent in any diffusion equation with a finite Kramers–Moyal expansion associated to a classical stochastic process, are intimately related to genuinely quantum effects due to the nonlinearities of the potential and imply that even a reduced Wigner function which is initially positive everywhere will acquire negative values when it evolves in time. Moreover, this higher derivative term would be uniquely responsible for tunneling if the system were isolated. Despite its remarkable features, this term has been neglected in this paper, since otherwise we were unable to obtain analytical results for the decay rate. This approximation, whose justification will be further discussed below, constitutes the most drastic assumption made throughout the paper.

Having neglected the higher derivative terms, the master equation is equivalent to the Fokker-Planck equation associated to a Langevin equation with a Gaussian stochastic source characterized by a nonlocal correlation function (the nonlocal noise kernel). From this point on, most of the approximations employed to compute the decay rate are more or less standard (Hänggi *et al.*, 1990). First, we change to action-angle variables and make use of an adiabatic approximation to eliminate the fast variable (the angle). This is consistent provided that $\gamma \ll \Omega_0$, which is in agreement with the previous assumption of small enough values for γ. Next, a harmonic approximation is introduced for the solutions to the equations of motion for the isolated system which appear in both the master equation and the Fokker–Planck equation. This approximation helps to obtain a rather simple result for the decay rate and can be justified both qualitatively and quantitatively, in contrast to the field theory case analyzed by Calzetta *et al.* (2001, 2002), where the existence of a threshold for the dissipation and noise kernels would preclude such an approximation. Finally, it is assumed that the characteristic decay time is

[7] One might be concerned that the truncation of higher orders in γ when considering times much larger than the characteristic relaxation timescale (i.e., $\gamma t \gg 1$), as required to compute the decay rate, might no longer be valid because of the existence of secular terms among the terms of higher order in γ that have been neglected. Although arguments that justify such a truncation when computing the decay rate can be given, this point might deserve a more careful analysis.

much smaller than the relaxation and dynamical timescales: $r \ll \gamma \ll \Omega_0$, where r is the decay rate (the inverse of the decay time). Although γ was required to be small enough, the previous inequality can be fulfilled provided that the potential barrier is sufficiently high, i.e., ε_s is large enough.

After reviewing the main approximations employed, let us compare the result obtained for the decay rate to that of Caldeira and Leggett (1981, 1983b) as well as to the tunneling rate when the coupling to the environment is absent. Whereas Caldeira and Leggett found that the interaction with the environment tends to suppress tunneling, we are not going to recover that result since, having neglected the higher derivative terms responsible for tunneling in an isolated system, the dissipation and diffusion terms that appear in the master equation due to the backreaction of the environment will always lead to a positive (or, at most, vanishing) probability of escaping from the metastable well. Nevertheless, if the decay rate obtained were much smaller or much larger than the tunneling rate for the isolated system so that the timescales governing both processes are very different, one could expect that the contribution to the decay rate from the process with a shortest characteristic timescale would be dominant.

The tunneling rate for an isolated system initially trapped in the metastable minimum of the potential considered in this paper is (Caldeira and Leggett, 1983b)

$$r_t \sim \Omega_0 \left(\frac{\varepsilon_s}{\varepsilon_0}\right)^{1/2} \exp\left(-\frac{18}{5}\frac{\varepsilon_s}{\varepsilon_0}\right), \tag{56}$$

where we recall that ε_0 is the zero-point energy of a harmonic oscillator of frequency Ω_0. On the other hand, Caldeira and Leggett (1981, 1983a) obtained the modification of the tunneling rate due to the interaction with the environment, which in the case of small dissipation is given by

$$r_t \sim \Omega_0 \left(\frac{\varepsilon_s}{\varepsilon_0}\right)^{1/2} \exp\left[-\frac{\varepsilon_s}{\varepsilon_0}\left(\frac{18}{5} + \frac{54\zeta(3)}{\pi^3}\frac{\gamma}{\Omega_0}\right)\right]. \tag{57}$$

Therefore, since the interaction with the environment simply adds a negative contribution to the exponent, it always tends to suppress the tunneling rate. Finally, the decay rate due to the activation-like effect obtained in the previous section corresponds to

$$r \sim \gamma \frac{\varepsilon_s}{\varepsilon_0} \exp\left(-\frac{\varepsilon_s}{\varepsilon_0}\right), \tag{58}$$

which is valid for small γ and large γ_s. When γ is very small, the decay rates from Eqs. (56) and (57) become very close. Furthermore, although a small γ implies a small contribution to the decay rate associated to the activation-like process, it can always be made arbitrarily larger than the tunneling rate by taking ε_s large enough.

The fact that the activation-like decay rate can be made arbitrarily large as compared to the tunneling rate for the isolated system seems to suggest, as

mentioned earlier, that even if the two phenomena were considered simultaneously, the former would be expected to dominate. If that were the case, it would imply that the usual instanton methods, when applied to a system interacting with an environment, downplay the role of the backreaction of the environment on the system dynamics. This can be qualitatively interpreted in the following way: while, roughly speaking, the tunneling effect for an isolated system can be regarded as a consequence of the energy fluctuations implied by Heisenberg's uncertainty principle, the interaction with an environment would induce fluctuations on the system due to the quantum fluctuations of the environment itself (Nagaev and Büttiker, 2002), which would enhance the tunneling rate.

Nevertheless, to reach a definite conclusion, it would be desirable to deal with the two contributions simultaneously and make sure that the higher derivative terms do not disrupt the effect of the backreaction terms, even when the timescale for the contribution to the decay rate from the former terms is much longer than the timescale associated to the activation-like process. Unfortunately, dealing with the higher derivative terms is not an easy task, and it seems hard to provide a real-time description of the tunneling process in terms of the Wigner function even for an isolated system (see, however, Risken and Vogel, 1988, for a first step in that direction). One may try to gain partial information on the relation between the two processes by considering different potentials with wider or narrower barriers, since one would naively expect that tunneling is suppressed for wide barriers while the activation-like contribution does not depend so much on the width of the barrier, as long as the height remains the same. A couple of comments concerning the freedom to modify the system potential are, nevertheless, in order. First, the potential must be analytic in order to derive the master equation for the reduced Wigner function. Second, when solving the formally equivalent problem of a classical Brownian particle escaping from the potential well, we assumed that once the particle reaches the maximum of the barrier it escapes and never comes back, but if a very wide barrier is considered, the probability that the particle comes back because of the fluctuations may become no longer negligible.

We close this section insisting on the importance of finding a satisfactory method to deal with the higher derivative terms, which would be very helpful to elucidate whether the enhancement of the decay rate obtained in this paper and entirely due to the backreaction from the environment fluctuations would still persist when the terms responsible for tunneling in isolated systems are also taken into account. Such a method would have an interest in its own right even if the results of Caldeira and Leggett (1981, 1983b) were finally recovered when properly taking into account the higher derivative terms, since it would constitute a key step in formulating a real-time description of tunneling.

ACKNOWLEDGMENTS

We thank Bei-Lok Hu, Fernando Lombardo, Slava Mukhanov, and Renaud Parentani for interesting discussions in Peyresq. D.A. acknowledges support of an FI grant from the Generalitat de Catalunya, and A.R. acknowledges support from NSF under Grant No. PHY-9800967. This work is also supported in part by MICYT Research Project No. FPA-2001-3598, European Project HPRN-CT-2000-00131, ANPCYT through Project No. PICT99 03-05229, CONICET, UBA, and Fundacion Antorchas.

REFERENCES

Caldeira, A. O. and Leggett, A. J. (1981). Influence of dissipation on quantum tunneling in macroscopic systems. *Physical Review Letters* **46**, 211–214.

Caldeira, A. O. and Leggett, A. J. (1983a). Path integral approach to quantum Brownian motion. *Physica* **121A**, 587–616.

Caldeira, A. O. and Leggett, A. J. (1983b). Quantum tunnelling in a dissipative system. *Annals of Physics* **149**, 374–456.

Callan, C. G. and Coleman, S. R. (1977). The fate of the false vacuum: II. First quantum corrections *Physical Review D: Particles and Fields* **16**, 1762–1768.

Calzetta, E., Roura, A., and Verdaguer, E. (2001). Vacuum decay in quantum field theory. *Physical Review D: Particles and Fields* **64**, 105008.

Calzetta, E., Roura, A., and Verdaguer, E. (2002). Dissipation, noise and vacuum decay in quantum field theory. *Physical Review Letters* **88**, 010403.

Calzetta, E., Roura, A., and Verdaguer, E. (2003). Stochastic description for open quantum systems. *Physica* **319A**, 188–212.

Calzetta, E. and Verdaguer, E. (1999). Noise induced transitions in semiclassical cosmology. *Physical Review D: Particles and Fields* **59**, 083513.

Coleman, S. R. (1977). The fate of the false vacuum: I. Semiclassical theory. *Physical Review D: Particles and Fields* **15**, 2929–2936.

Coleman, S. R. (1985). *Aspects of Symmetry*, Cambridge University Press, Cambridge, England.

Courant, R. and Hilbert, D. (1953). *Methods of Mathematical Physics*, Interscience, New York.

Feynman, R. P. and Hibbs, A. (1965). *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York.

Feynman, R. P. and Vernon, F. L. (1963). The theory of a general quantum system interactiong with a linear dissipative system. *Annals of Physics* **24**, 118–173.

Fujikawa, K., Iso, S., Sasaki, M., and Suzuki, H. (1992a). Canonical formulation of quantum tunneling with dissipation. *Physical Review Letters* **68**, 1093–1096.

Fujikawa, K., Iso, S., Sasaki, M., and Suzuki, H. (1992b). Quantum tunneling with dissipation: Possible enhancement by dissipative interactions. *Physical Review B: Condensed Matter* **46**, 10295–10309.

Galindo, A. and Pascual, P. (1991). *Quantum Mechanics*, Springer-Verlag, New York.

Goldstein, H. (1980). *Classical Mechanics*, Addison-Wesley, Reading, MA.

Gradsteyn, I. S. and Ryzhik, I. M. (1980). *Tables of Integrals, Series, and Products*, Academic Press, New York.

Hänggi, P., Talkner, P., and Borkovec, M. (1990). Reaction-rate theory: Fifty years after Kramers. *Reviews of Modern Physics* **62**, 251–341.

Hillery, M., O'Connell, R. F., Scully, M. O., and Wigner, E. P. (1984). Distribution functions in physics: Fundamentals. *Physics Reports* **106**, 121–167.

Kramers, H. A. (1940). Brownian motion in a field of force and the diffusion model of chemical reactions. *Physica (Amsterdam)* **7**, 284–304.

Landau, L. D. and Lifshitz, E. M. (1977). *Quantum Mechanics (Non Relativistic Theory)*, Pergamon, London.

Nagaev, K. E. and Büttiker, M. (2002). Ground state energy fluctuations of a system coupled to a bath. *Europhysics Letters* **58**, 475–481.

Risken, H. and Vogel, K. (1988). Quantum treatment of dispersive optical bistability. In *Lecture Notes of the X Sitges Summerschool on "Far From Equilibrium Phase Transitions*," Springer-Verlag, Berlin, pp. 237–262.

Roura, A. (2001). *Stochastic Description for Open Quantum Systems and Cosmological, Perturbations*, PhD Thesis, Universitat de Barcelona.

Roura, A. and Verdaguer, E. (1999). Mode decomposition and renormalization in semiclassical gravity. *Physical Review D: Particles and Fields* **60**, 107503.

Schwartz, L. (1957). *Théoric des distributions*, 2 vols., Hermann, Paris.

Wigner, E. P. (1932). Quantum corrections for thermodynamic equilibrium. *Physical Review* **40**, 749–760.